Dynamics and delocalization transition for an interface driven by a uniform shear flow

LETTER TO THE EDITOR

Dynamics and delocalization transition for an interface driven by a uniform shear flow

Rui D M Travasso, Alan J Bray and Andrea Cavagna

Department of Physics and Astronomy, University of Manchester, Manchester M13 9PL, UK

Abstract
We study the effect of a uniform shear flow on an interface separating the two broken-symmetry ordered phases of a two-dimensional system with non-conserved scalar order parameter. The interface, initially flat and perpendicular to the flow, is distorted by the shear flow. We show that there is a critical shear rate, $\gamma_c \propto 1/L^2$ (where L is the system width perpendicular to the flow), below which the interface can sustain the shear. In this regime the countermotion of the interface under its curvature balances the shear flow, and the stretched interface stabilizes into a time-independent shape whose form we determine analytically. For $\gamma > \gamma_c$ the interface acquires a non-zero velocity, whose profile is shown to reach a time-independent limit which we determine exactly. The analytical results are checked by numerical integration of the equations of motion.

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(Some figures in this article are in colour only in the electronic version; see www.iop.org)

There is growing interest in understanding the effects of shear on the dynamical properties of statistical systems far from equilibrium [1]. The most extensively studied case is perhaps the approach to equilibrium of a system quenched below its critical point: domain growth is heavily influenced by the external shear and new dynamical exponents appear [2]. The determination of these exponents, together with the problem of the validity of dynamical scaling, are the most challenging tasks in this context [3].

　　The transition from a disordered, high-temperature phase to an ordered, low-temperature one, however, is not the only context where an applied shear may play a major role. An interesting alternative problem is to investigate cases where the shear may by itself introduce a novel dynamics in a state otherwise ordered and stable. In the case of spinodal decomposition of a binary fluid, for example, we may let the system evolve until a stable, entirely separated state is reached, and then apply a shear flow normal to the interface between the two phases. Further evolution will then occur, with a competition between the shear, which tries to stretch the interface, and diffusion of the two constituents, which tends to straighten it.

More generally, a natural question in this context is to what extent the applied shear is able to perturb the stable initial state. More specifically, we may ask: is there a critical value for the intensity of the shear, beyond which the system is unable to restore itself to a stable stationary state?

In this Letter we shed some light on the above questions by studying the dynamics of a flat interface subjected to a transverse shear flow. The motion of such an interface in the case of conserved dynamics has been recently studied in [4]. For the shear rates studied, the stretched interface always reached a stable steady state. In this Letter, we study the deterministic dynamics of a similar interface under shear, but for the case of non-conserved dynamics, in a system described by a scalar order parameter. Ising-like systems, such as twisted nematic liquid crystals [5], display a behaviour that can be described by this model. For weak shear we find, similarly to [4], that the interface reaches a stationary profile, in which the curvature forces acting on the interface compensate the shear. In this way the interface slips relative to the moving boundaries, and acquires a steady-state profile with no net velocity in the laboratory frame. However, for shear rates larger than a critical value we find that the interface cannot sustain the strain and a time-independent state can no longer be reached. In this regime, the interface departs indefinitely from its initial condition and becomes delocalized. Instead of a stationary *spatial* profile, the interface acquires a stationary *velocity* profile. It should be noted that the speed of the interface at the boundaries is always smaller than that of the boundaries themselves, i.e. the contact points still slip with respect to the moving boundaries, but not enough to keep the interface stationary in the laboratory frame.

The system we study is a two-dimensional strip, bounded in the y direction between the values $-y_0$ and y_0, so its width is $L = 2y_0$. We consider a uniform shear flow, given by the simple velocity profile $v = \gamma y e_x$, where γ is the shear rate and e_x is a unit vector in the x-direction, which is thus the direction of the flow. At time $t = 0$, the interface is given by the equation $x = 0$, i.e. it is a flat segment connecting the boundaries and separating two regions with opposite value of the order parameter. We will consider here only the zero temperature dynamics of such a system.

As a physical boundary condition, we assume that the interface is perpendicular to the boundaries at the points of contact: any other condition would create an infinite force at such points due to the uneven curvature. We can understand this argument better in the context of the Ising model, where the effect of a shear is simulated by simultaneously moving every row of spins parallel to the boundaries by one lattice spacing, relative to the row below, at time intervals $\Delta t = 1/\gamma$, corresponding to a shear rate γ. Consider an interface in this system which is *not* normal to the boundary, such that one part of the boundary makes an acute angle with the interface at the contact point, while the neighbouring part of the boundary, on the other side of the contact point, makes an obtuse angle with the interface. The spin located within the acute angle, next to the contact point, has an excess of neighbouring spins with opposite sign, and it therefore flips. This makes the interface become normal to the boundary. This argument only holds if the microscopic flipping time τ_0 is smaller than the time, $1/\gamma$, needed by the 'flow' to move the spin one lattice spacing relative to spins in the neighbouring row, that is if $\gamma < 1/\tau_0$. We will find a critical value of the shear $\gamma_c \ll 1/\tau_0$ and therefore we can consistently assume an interface normal to the boundaries at the contact points. Moreover, we have checked that an Ising simulation with free boundary conditions produces an interface which is indeed normal to the boundaries at the contact points.

The deterministic dynamics of a non-conserved order parameter, $\phi(x, t)$, is described by the time-dependent Ginzburg–Landau equation [6]

$$\tau_0 \frac{\partial \phi}{\partial t} = \xi_0^2 \nabla^2 \phi - V'(\phi) \tag{1}$$

where τ_0 is the relaxation time for order parameter fluctuations in the bulk, ξ_0 is the interfacial width, and $V(\phi)$ is a symmetric double well potential. Starting from equation (1) one can show using standard methods that the normal velocity of an interface separating the two phases is proportional to the local curvature, K. This is the Allen–Cahn equation [7], $v = -D\,K$, where $D = \xi_0^2/\tau_0$ is the diffusion constant for fluctuations in the bulk phases, and the curvature is $K = \nabla \cdot n$, where n is the local normal to the interface. The Allen–Cahn equation is actually more general than the specific dynamical equation (1), since it simply expresses the dynamics of an interface driven by its surface tension.

In the absence of shear, a flat interface is stable. When a shear is applied, however, the Allen–Cahn equation has to be modified:

$$v = -D\,K + \gamma y\,e_x \cdot n \tag{2}$$

where the additional term represents the advection of the interface by the flow, leading to a distortion of its initially planar form. Describing the interface profile by the function $x(y, t)$, which gives the displacement in the flow direction as a function of the position in the direction transverse to the flow, the intrinsic curvature of the function $x(y, t)$ is given by the standard result

$$K = -\frac{\partial_{yy}x}{[1 + (\partial_y x)^2]^{3/2}} \tag{3}$$

while the velocity, v, of the interface can be written as

$$v = \frac{\partial_t x}{[1 + (\partial_y x)^2]^{1/2}} \tag{4}$$

where $\partial_t x$ is the velocity of the interface in the direction parallel to the flow. Combining (2)–(4) we obtain an equation for the interface profile $x(y, t)$:

$$\partial_t x = D\,\frac{\partial_{yy}x}{1 + (\partial_y x)^2} + \gamma y. \tag{5}$$

Prior to further analysis, it is convenient to rescale space and time,

$$X \equiv x/y_0 \qquad Y \equiv y/y_0 \qquad \tau \equiv t\,D/y_0^2 \tag{6}$$

where now $Y \in [-1, 1]$. Note that in this way we are also introducing a rescaled interface velocity $V = v\,y_0/D$. In terms of the rescaled variables, equation (5) reads

$$\partial_\tau X = \frac{\partial_{YY}X}{1 + (\partial_Y X)^2} + \alpha Y \qquad \alpha \equiv \gamma y_0^2/D. \tag{7}$$

Due to the rescaling, all the dimensionfull parameters have been absorbed into the dimensionless effective shear rate α. Given the geometry of the problem, we clearly expect $X(Y, \tau)$ to be an odd function of Y. We will therefore limit our analysis to the domain $Y \in [0, 1]$.

In order to find a stationary solution $X_s(Y)$ of equation (7), we set $\partial_\tau X = 0$, to obtain (where primes indicate derivatives with respect to Y)
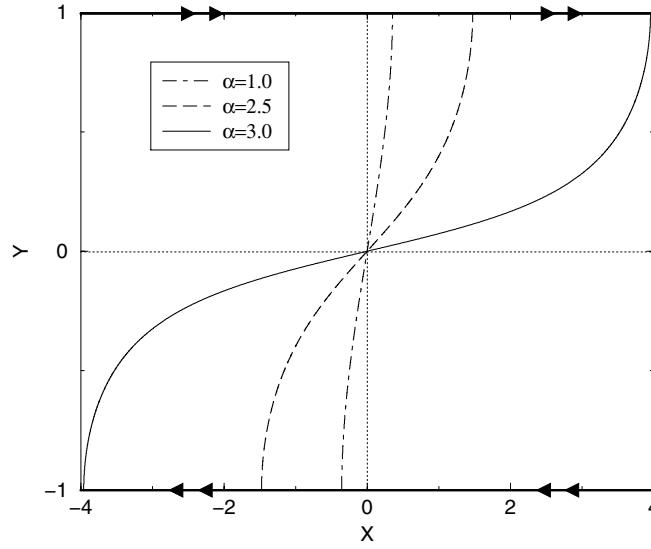
$$\frac{X_s''}{1 + (X_s')^2} = -\alpha Y \tag{8}$$

with boundary conditions

$$X_s'(1) = 0 = X_s(0) \tag{9}$$

where the first condition follows from the requirement that the interface be perpendicular to the boundaries, and the second from the fact that $X(Y)$ is an odd function. Integrating once, and imposing the boundary condition at $Y = 1$, gives

$$X_s'(Y) = \tan\left[\frac{\alpha}{2}(1 - Y^2)\right]. \tag{10}$$

Figure 1. Stationary interface profile as given by equation (11), for three different values of the effective shear rate α. The initial configuration of the interface is given by the flat $X = 0$ segment.

A second integration, incorporating the boundary condition at $Y = 0$, gives the stationary interface profile,

$$X(Y) = \int_0^Y \tan\left[\frac{\alpha}{2}(1 - z^2)\right] dz. \tag{11}$$

This interface profile is plotted in figure 1 for different values of the effective shear rate.

From equation (10) we see that the function $X_s(Y)$ has a maximum in its derivative at $Y = 0$. When $\alpha = \pi$ the derivative diverges at this point, indicating that the interface is parallel to the system boundaries. A value $\alpha > \pi$ would imply a discontinuous derivative $X_s'(Y)$ and thus an interface profile $X_s(Y)$ with two cusps: such a profile would be unphysical, because of the infinite curvature at the cusps. We conclude that for $\alpha > \pi$ no stationary solution is possible. This implies that there is a critical value of the dimensionless shear rate,

$$\alpha_c = \pi \tag{12}$$

i.e. a critical shear rate

$$\gamma_c = \frac{4\pi D}{L^2}. \tag{13}$$

For $\alpha > \alpha_c$ (or $\gamma > \gamma_c$) we must find a time-dependent solution of the full equation (7). In terms of the microscopic parameters of the system, we have

$$\gamma_c = \frac{4\pi \xi_0^2}{L^2} \frac{1}{\tau_0} \tag{14}$$

and therefore $\gamma_c \ll 1/\tau_0$ as long as the interfacial width is much smaller than the size of the system, a condition that obviously holds for any reasonable system. Therefore, as anticipated in the introduction, our assumption of an interface normal to the boundaries at the contact points is fully justified in the totality of the interesting γ regime.

For $\alpha \to \alpha_c$ the contact point $X(1)$ goes to infinity and the interface is no longer localized in a finite region of space. We have

$$X(1) \sim \sqrt{\frac{\alpha_c}{\alpha - \alpha_c}} \qquad \alpha \to \alpha_c. \tag{15}$$
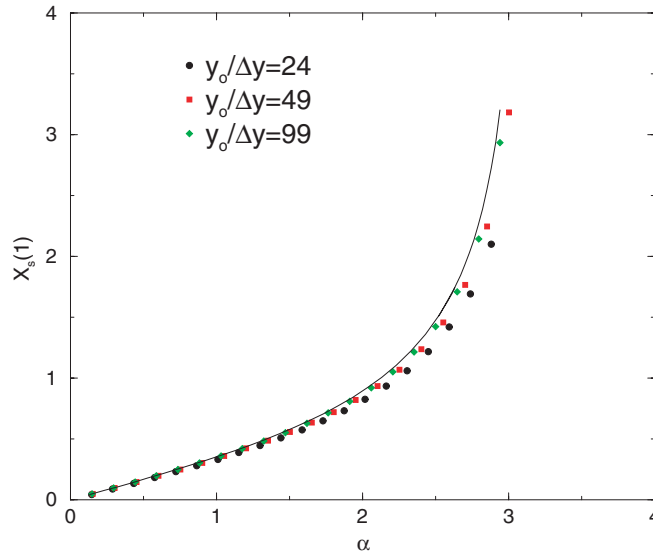
Figure 2. Numerical integration of equation (7) (symbols) and analytical result (full curve). The numerical integration approaches the analytical result as the spatial discretization scale, Δy, decreases.

In order to obtain this result it must be noted that for $\alpha \to \alpha_c$, the integral in equation (11) is completely concentrated near the origin, $z = 0$, and it therefore depends very weakly on the value of Y, as long as Y is non-vanishing. This means that for $\alpha \to \alpha_c$ not only the contact point $X(1)$, but *all* the points of the interface $X(Y)$ with non-zero Y are found at the same position given by (15).

Before considering the $\alpha > \alpha_c$ regime, we want to check that the time-independent solution we have found is indeed an attractor for the interface dynamics, when starting from the initial configuration $X(Y, 0) = 0$. To do this, equation (7) was discretized and integrated numerically. In figure 2 we plot the position of the contact point, $X_s(1)$, of the asymptotic time-independent solution as a function of the shear rate, in order to compare our analytic result, equation (11), with the numerical integration of equation (7). The agreement is good, and it can be seen that it improves as the continuum limit is approached. The conclusion is that the stationary state given by equation (11) is indeed an attractor for the interface dynamics.

When $\alpha > \alpha_c$ the time-independent equation for the interface has no physical solution and the interface must move with a non-zero velocity parallel to the boundaries. A reasonable assumption is that, in the large-time limit, the velocity of every point along the interface is time independent, i.e. $\partial_\tau X(Y, \tau) \equiv V(Y, \tau) \to V_\infty(Y)$. Such a velocity profile for the interface implies that the nonlinear term in equation (7) must be time-independent in the asymptotic limit. This condition must be satisfied in two different ways, according to the different regions of the Y-domain.

- $Y \sim 0$. From the time-independent equation (8) we see that $X_s''(0) = 0$ for all $\alpha < \alpha_c$. It is natural then to assume that even in the phase $\alpha > \alpha_c$ the interface is asymptotically flat close to the centre, such that the nonlinear term vanishes in this region and the solution of the equation is trivial:

$$X(Y, \tau) = \alpha Y \tau. \tag{16}$$

This solution clearly satisfies equation (7), and the boundary condition $X(0, \tau) = 0$. However, it cannot be correct close to the boundary at $Y = 1$ because it does not satify the boundary condition $\partial_Y X(1, \tau) = 0$. We will find that it is correct in a domain $0 \leqslant Y \leqslant Y_1$, with Y_1 to be determined below.

- $Y \sim 1$. At $Y = 1$, the interface must satisfy the boundary condition $\partial_Y X(1, \tau) = 0$, and therefore we need to keep the nonlinear term in the equation of motion. However, as we already noted, this term must become independent of time for large times. This can be achieved by requiring that

$$X(Y, \tau) \equiv f(Y) + V_0\tau \qquad Y \sim 1 \tag{17}$$

so that the portion of the interface close to the system boundary moves with a velocity V_0 independent of Y. This solution also satisfies equation (7). We shall find that it is correct in the domain $Y_1 < Y \leqslant 1$. Integrating equation (7) with $\partial_\tau X(Y, \tau) = V_0$ and $\partial_Y X(1, \tau) = 0$, we obtain

$$\partial_Y X = \tan\left[\frac{\alpha}{2}(1 - Y^2) - V_0(1 - Y)\right]. \tag{18}$$

The two solutions, equations (16) and (18), must be matched at the border between the two different regions. Let us define region I: $0 \leqslant Y \leqslant Y_1$ and region II: $Y_1 \leqslant Y \leqslant 1$. The matching point Y_1 will be calculated later. We can write the complete interface velocity profile, for $\tau \to \infty$, as

$$V_\infty(Y) = \begin{cases} \alpha Y & \text{I} \\ V_0 = \alpha Y_1 & \text{II.} \end{cases} \tag{19}$$

For the Y derivative of the interface profile, for $\tau \to \infty$, we have

$$\partial_Y X_\infty(Y) = \begin{cases} \infty & \text{I} \\ \tan\left[\frac{\alpha}{2}(1 - Y^2) - V_0(1 - Y)\right] & \text{II.} \end{cases} \tag{20}$$

Furthermore, note that the value of V_0 is known, from equation (19), once Y_1 is determined. In order to calculate Y_1 we have to match the solutions in the different regions at this point. Note that the derivative, $\partial_Y X$, has to be continuous at this point to avoid an infinite curvature. From (20), we see that this requires the argument of the tangent in region II to be equal to $\pi/2$ when $Y = Y_1$. This gives

$$\begin{aligned} Y_1 &= 1 - \sqrt{\frac{\alpha_c}{\alpha}} \\ V_0 &= \alpha\left(1 - \sqrt{\frac{\alpha_c}{\alpha}}\right). \end{aligned} \tag{21}$$

Note that the velocity V_0 of the interface at the contact point, $Y = 1$, is smaller than the velocity of the flow at this same point which, in our dimensionless variables, is equal to α. Our analytical solution was tested by numerical integration of the dynamical equation, and the results are shown in figure 3. Numerical and analytical results are fully consistent.

The convergence of the interface velocity at the contact point, $\partial_\tau X(1, \tau) \to V_0$ for $\tau \to \infty$, was found to be very slow. The function $\partial_\tau X(1, t)$ is monotonically decreasing in time: initially it is equal to the flow velocity at the contact point α, but eventually it must decrease, in order to develop a curvature to keep the interface perpendicular to the boundary. A closer inspection of figure 3 shows that for any finite time the cusp in the velocity profile is rounded out and this function is smooth.

However, for very large times, it is not unreasonable to approximate the actual velocity profile with the same piecewise linear form as in equation (20) (see figure 4). Of course, we
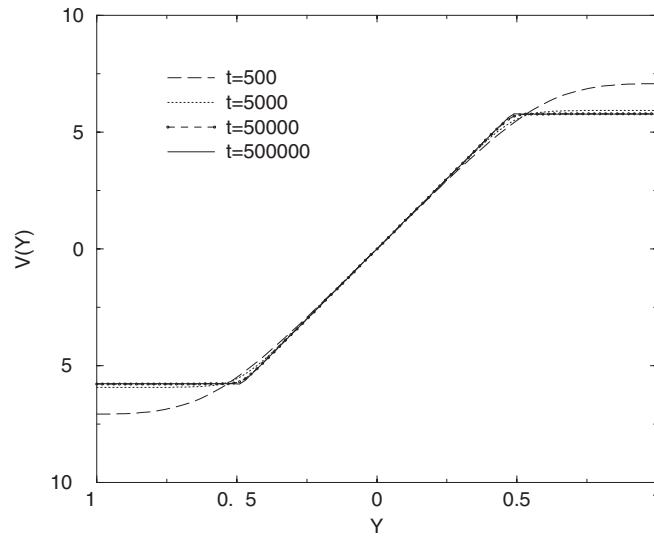
Figure 3. The velocity profile of the interface as a function of Y, with $\alpha = 12$. It can be clearly seen that for large times the velocity profile converges towards the function $V_\infty(Y)$ of equation (19).
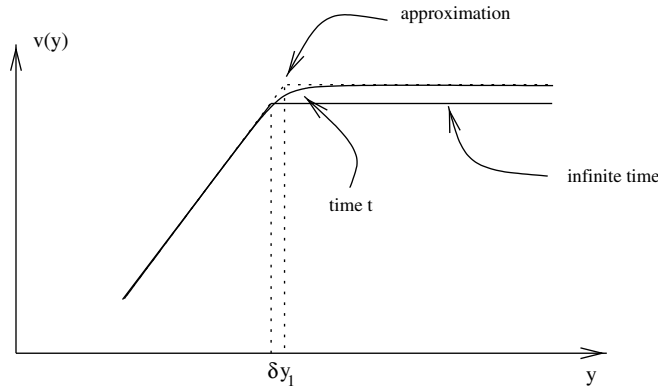
Figure 4. The nature of the approximation leading to equation (22).

have to introduce a time-dependent value, $Y_1(\tau)$, for the matching point, with $Y_1(\tau) \to Y_1$ for $\tau \to \infty$. Within this approximation we can therefore write

$$\partial_Y X(Y, \tau) \approx \begin{cases} \alpha\tau & \text{I} \\ \tan\left[\dfrac{\alpha}{2}(1 - Y^2) - V_0(\tau)(1 - Y)\right] & \text{II} \end{cases} \tag{22}$$
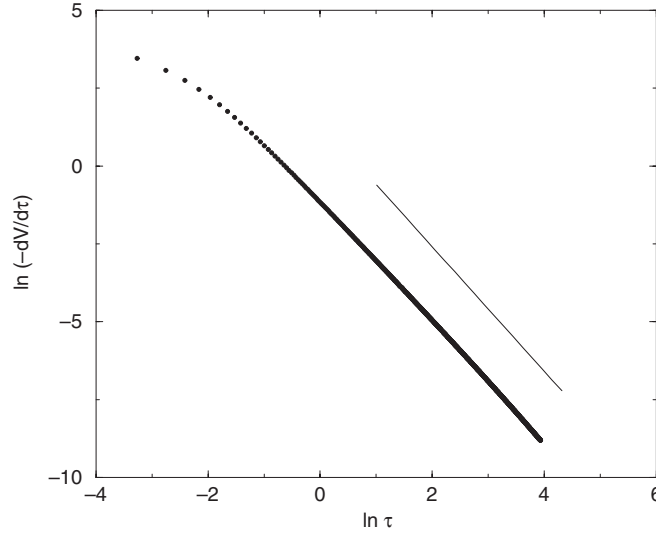
where $V_0(\tau) = \alpha Y_1(\tau)$ and $Y_1(\tau) = Y_1 + \delta Y_1(\tau)$. The new value of the matching point $Y_1(\tau)$ is fixed, as usual, by imposing the continuity of $\partial_Y X(Y, \tau)$. This gives

$$\tan^{-1}(\alpha\tau) \approx \frac{\pi}{2} - \frac{1}{\alpha\tau} \approx \frac{\alpha}{2}[1 - Y_1(\tau)]^2. \tag{23}$$

Setting $Y_1(\tau) = Y_1 + \delta Y_1$ and recalling that $\frac{\alpha}{2}(1 - Y_1)^2 = \frac{\pi}{2}$ from the definition of Y_1 in equation (21), we obtain

$$\delta Y_1 \approx \frac{1}{\alpha^2(1 - Y_1)} \frac{1}{\tau} \tag{24}$$

Figure 5. The quantity $\partial_\tau V(1, \tau)$ as a function of τ in a log–log plot. The full line has slope -2, for a comparison with the analytical result of equation (26).

and for $V(1, \tau)$

$$V(1, \tau) \approx V_0 + \frac{1}{\sqrt{\alpha \alpha_c}} \frac{1}{\tau}. \tag{25}$$

In order to check this last result numerically, it is convenient to compute the time derivative of $V(1, \tau)$, which does not contain the constant V_0. We have

$$\partial_\tau V(1, \tau) \approx -\frac{1}{\sqrt{\alpha \alpha_c}} \frac{1}{\tau^2}. \tag{26}$$

This last result is tested in figure 5, where we may see that again numerical integration of the dynamical equation and analytical results coincide.

In this Letter we have analytically and numerically studied the zero temperature dynamics of an interface subject to a transverse shear flow, in the case of non-conserved dynamics. We find a critical value, $\alpha_c = \pi$, of the dimensionless shear rate, $\alpha = \gamma y_0^2/D$, beyond which a steady state cannot be reached and the interface moves with a constant velocity. In terms of the thickness, $L = 2y_0$, of the sample, this gives $\gamma_c = 4\pi D/L^2$, i.e. the transition occurs at lower shear rates for wider systems.

While no such critical value was reported for the conserved case analysed in [4], it is unclear to us why there should be a major difference in this respect between conserved and non-conserved dynamics. Indeed, in the latter case, the dynamics is slower, due to the conservation constraint, and the system should be less capable of sustaining the shear than in the non-conserved case. We conjecture, therefore, that a similar transition occurs for systems with conserved dynamics.

References

[1] See Onuki A 1997 *J. Phys.: Condens. Matter* **9** 6119 and references therein

[2] Chan C K, Perrot F and Beysens D 1991 *Phys. Rev.* A **43** 1826
     Hashimoto T, Matsuzaka K, Moses E and Onuki A 1995 *Phys. Rev. Lett.* **74** 126
     Läuger J, Laubner C and Gronski W 1995 *Phys. Rev. Lett.* **75** 3576

[3] Rothman D H 1990 *Phys. Rev. Lett.* **65** 3305
     Wagner A J and Yeomans J M 1999 *Phys. Rev.* E **59** 4366

[4] Chen H, Jasnow D and Viñals J 2000 *Phys. Rev. Lett.* **85** 1686

[5] Orihara H and Ishibashi Y 1986 *J. Phys. Soc. Japan* **55** 2151

[6] Landau L D and Khalatnikov I M 1954 *Dokl. Akad. Nauk SSSR* **96** 469

[7] Allen S M and Cahn J W 1979 *Acta Metall.* **27** 1085
     for a review see Bray A J 1994 *Adv. Phys.* **43** 357 and references therein